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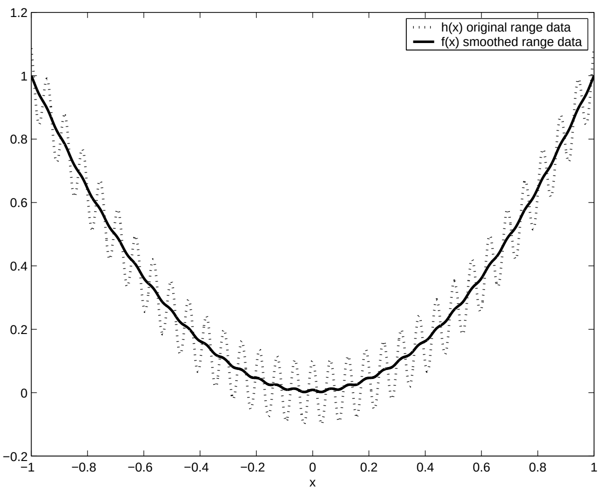
Visual Computing
Variational Calculus
Solution

General Remarks

It is not necessary to hand in your results. You will find an exemplary solution on the lecture's web page.

1) Data Smoothing (from WRK exam 04/05)

Consider a robot equipped with a one-dimensional laser range finder. The sensor provides measurements of range data $h(x)$ for a given interval $x \in [a, b]$. Since range data can be very noisy we like to smooth these measurements, i.e., we search for a function $f(x)$ which approximates as good as possible the measured data $h(x)$ while avoiding large gradients. The first objective of data faithfulness is achieved in the simplest version by penalizing quadratic deviations of $f(x)$ from data $h(x)$, while the smoothness objective is introduced by an additional term containing the integrated squared gradient f_x.



a) Derive the functional $I(f)$ which models these two objectives ('1D variational method'). Weight the smoothness term containing the squared gradient by the regularization parameter $1/\mu^2$ (this will simplify the solution).

Solution: The first objective of data faithfulness is achieved by finding a function $f(x)$ that minimizes quadratic deviations from the data $h(x)$:

$$\int_a^b (f(x) - h(x))^2 \ dx.$$

The additional smoothness objective is based on the integrated squared gradient:

$$\int_a^b (f_x(x))^2 \ dx.$$

Using the regularization parameter μ^{-2}, we obtain the following functional for smooth data interpolation:

$$I(f) = \int_a^b \underbrace{\left((f-h)^2 + \mu^{-2}(f_x)^2\right)}_{F(x,f,f_x)} \ dx.$$

The parameter μ can be used to adjust the desired smoothness of the solution f.

b) Which necessary condition has to hold for the function f minimizing the smoothing costs $I(f)$ obtained in a)?

Solution: The minimizing function f has to satisfy the Euler-Lagrange equation $\frac{\partial F}{\partial f} = \frac{d}{dx}\left(\frac{\partial F}{\partial f_x}\right)$. In our case, we obtain:

$$\frac{\partial F}{\partial f} = 2(f-h), \qquad \frac{d}{dx}\left(\frac{\partial F}{\partial f_x}\right) = 2\mu^{-2} f_{xx},$$

which results in the linear ordinary differential equation

$$f_{xx} = \mu^2(f-h).$$

c) We will now consider a parabolic wall with cosine modulation in depth, i.e., the range data is given for $x \in [-1, 1]$ by

$$h(x) = x^2 + \epsilon\cos(\omega x),$$

where ϵ and ω can be assumed to be fixed parameters. In this case, the necessary condition obtained in b) has a general solution of the form

$$f(x) = c_1\exp(\mu x) + c_2\exp(-\mu x) + c_3\cos(c_4 x) + c_5 + c_6 x^2,$$

where c_i, $1 \le i \le 6$ are free parameters. Verify that the solution has this functional form by determining parameters $c_1, \ldots, c_6$ which meet the boundary constraints $f(-1) = f(1) = 1$.

Hint: Group similar terms and determine the parameters by comparing the coefficients.

Solution: Partial derivatives for the given general solution:

$$\begin{aligned} f_x(x) &= c_1\mu\exp(\mu x) - c_2\mu\exp(-\mu x) - c_3 c_4\sin(c_4 x) + 2c_6 x \\ f_{xx}(x) &= c_1\mu^2\exp(\mu x) + c_2\mu^2\exp(-\mu x) - c_3 c_4^2\cos(c_4 x) + 2c_6. \end{aligned}$$

With the given range data function h, the Euler-Lagrange equation becomes

$$\begin{aligned} &c_1\mu^2\exp(\mu x) + c_2\mu^2\exp(-\mu x) - c_3 c_4^2\cos(c_4 x) + 2c_6 \\ &\quad = \mu^2\left((c_1\exp(\mu x) + c_2\exp(-\mu x) + c_3\cos(c_4 x) + c_5 + c_6 x^2) - (x^2 + \epsilon\cos(\omega x))\right) \\ &\quad = c_1\mu^2\exp(\mu x) + c_2\mu^2\exp(-\mu x) + c_3\mu^2\cos(c_4 x) - \epsilon\mu^2\cos(\omega x) + c_5\mu^2 + (c_6 - 1)\mu^2 x^2. \end{aligned}$$

Comparing the coefficients of similar terms yields the following results:

$$\begin{aligned}
x^2\text{-}\textit{term}: &\quad c_6 = 1 \\
\textit{const. terms}: &\quad c_5 = \frac{2c_6}{\mu^2} = 2\mu^{-2} \\
\cos\text{-}\textit{argument}: &\ c_4 = \omega \\
\cos\text{-}\textit{terms}: &\ c_3\omega^2\cos(\omega x) = (\epsilon\mu^2 - c_3\mu^2)\cos(\omega x) \quad \Rightarrow \quad c_3 = \frac{\epsilon}{1+\frac{\omega^2}{\mu^2}}
\end{aligned}$$

The remaining parameters c_1 and c_2 are obtained from the boundary conditions $f(-1) = f(1) = 1$:

$$\begin{aligned}
1 &= c_1\exp(-\mu) + c_2\exp(\mu) + c_3\cos(-\omega) + 2\mu^{-2} + 1 \\
&= c_1\exp(\mu) + c_2\exp(-\mu) + c_3\cos(\omega) + 2\mu^{-2} + 1,
\end{aligned}$$

which gives:

$$\begin{aligned}
c_2 &= c_1 \\
c_1 &= -\left(\frac{\epsilon}{1+\frac{\omega^2}{\mu^2}}\cos\omega + \frac{2}{\mu^2}\right)\frac{1}{2\cosh(\mu)},
\end{aligned}$$

where we used the definition $\cosh(\mu) = \frac{1}{2}(\exp(\mu) + \exp(-\mu))$.

d) Robust smoothing

The quadratic penality for large gradients is often too strict. Replace the quadratic costs for both the data and the smoothness term by a 'robust' penalty $\log\cosh(\cdot)$ which is quadratic for small values and linear for large absolute values. Which necessary condition has to hold for this version of optimal robust smoothing?

Hint: The first and second derivatives of $g(x) = \log\cosh(x)$ are given by $g'(x) = \tanh(x)$ and $g''(x) = 1/\cosh^2(x)$.

Solution: The functional $I(f)$ for robust smoothing is given by

$$I(f) = \int_a^b \underbrace{\left(\log\cosh(f-h) + \mu^{-2}\log\cosh(f_x)\right)}_{F(x,f,f_x)}\, dx.$$

The derivatives

$$\frac{\partial F}{\partial f} = \tanh(f-h) \qquad \textit{and} \qquad \frac{d}{dx}\frac{\partial F}{\partial f_x} = \mu^{-2}\frac{d}{dx}\tanh(f_x) = \mu^{-2}\frac{f_{xx}}{\cosh^2(f_x)}$$

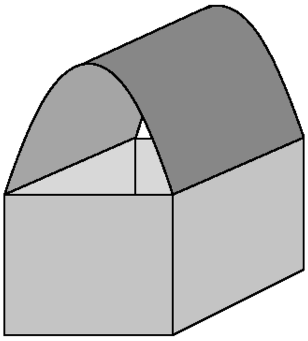
yield the Euler-Lagrange equation

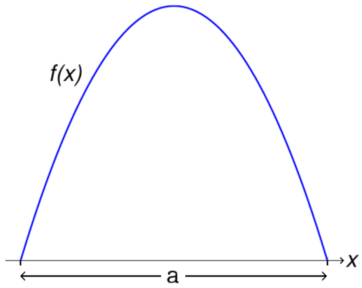
$$f_{xx} = \mu^2\cosh^2(f_x)\tanh(f-h).$$

This differential equation can be solved numerically by discretization.

2) An Isoperimetric Problem (from Mod. & Sim. exam 05/06)

Consider the following problem: Given a rectangular box of fixed size, we want to design a top cover with open ends that gives us as much room as possible (cf. figure). Due to bending constraints on the material of the top cover plate, its average squared gradient is restricted to a fixed value ρ^2. As the room under the top cover is given by the (fixed) length of the box multiplied with the area of the (open) end piece, it is sufficient to determine the optimal shape of the end piece. If we denote the fixed width of the box by a, this shape can be modeled by a function $f(x)$ in one dimension (cf. figure).





a) Derive the functional $I(f)$ that models the constrained objective by introducing the corresponding Lagrangian with a weight parameter λ. Attention: We are looking for a *maximum* value here.

Hint: Assume that the zeros of f are at $-\frac{a}{2}$ and $\frac{a}{2}$.

Solution: The objective is to maximize the area of the end piece:

$$\int_{-a/2}^{a/2} f(x)dx$$

under the boundary conditions $f(-\frac{a}{2}) = f(\frac{a}{2}) = 0$.

The constraint on the average squared gradient can be written as:

$$\frac{1}{a}\int_{-a/2}^{a/2} (f'(x))^2 dx = \rho^2.$$

Inverting the sign of the objective to obtain a minimization problem, and omitting constants, this results in the Lagrangian functional

$$I(f) = -\int_{-a/2}^{a/2} f(x)dx + \lambda \int_{-a/2}^{a/2} (f'(x))^2 dx.$$

b) Which necessary condition has to hold for the function f that optimizes the cost functional $I(f)$ obtained in a)?

Solution: Integrand: $F(x, f, f') = -f + \lambda(f')^2$ *with derivatives*

$$F_f = -1 \qquad\qquad F_{f'} = 2\lambda f'$$

Euler-Lagrange equation:

$$F_f = \frac{d}{dx}F_{f'} \qquad \Longleftrightarrow \qquad -1 = 2\lambda f''$$

c) Assume that the width of the box is given as $a = 2$. Derive the resulting solution of the equation obtained in b) as a function of the Lagrange parameter λ.

Solution: The Euler-Lagrange equation gives:

$$\begin{aligned} f''(x) &= -\frac{1}{2\lambda} \\ \Rightarrow \quad f'(x) &= -\frac{1}{2\lambda}x + c_1 \\ \Rightarrow \quad f(x) &= -\frac{1}{4\lambda}x^2 + c_1 x + c_2 \end{aligned}$$

Inserting the boundary conditions for $a = 2$:

$$\begin{aligned} f(-1) = 0: &\qquad -\frac{1}{4\lambda} - c_1 + c_2 = 0 \\ f(1) = 0: &\qquad -\frac{1}{4\lambda} + c_1 + c_2 = 0 \\ \Rightarrow &\quad c_1 = 0 \\ \Rightarrow &\quad c_2 = \frac{1}{4\lambda} \end{aligned}$$

Solution: $f(x) = \frac{1}{4\lambda}(-x^2 + 1)$

d) What is the final solution if you take into account that the average squared gradient of f is equal to ρ^2?

Solution: Gradient of f: $f'(x) = -\frac{1}{2\lambda}x$

Gradient constraint:

$$\frac{1}{a}\int_{-a/2}^{a/2}(f'(x))^2 dx = \rho^2 . \rho^2 = \frac{1}{2}\int_{-1}^{1}(f'(x))^2 dx = \frac{1}{2}\int_{-1}^{1}\frac{1}{4\lambda^2}x^2 dx = \left[\frac{1}{8\lambda^2}\frac{1}{3}x^3\right]_{-1}^{1} = \frac{1}{12\lambda^2}$$

$$\Rightarrow \quad \lambda = \frac{1}{2\sqrt{3}\rho}$$

Final solution:

$$f(x) = \frac{\sqrt{3}\rho}{2}(-x^2 + 1)$$

e) Now consider the 'inverse problem': For a fixed volume V under the top cover we look for the 'smoothest' shape of the cover, where smoothness is defined by a small average squared gradient. How does this setting change the shape of the general solution function f? (You don't need to calculate the solution, but you should justify your decision.)

Solution: 'Inverse problem':

$$I(f) = \frac{1}{a}\int_{-a/2}^{a/2}(f'(x))^2 dx + \lambda\int_{-a/2}^{a/2} f(x)dx$$

Substituting $g = -\lambda f$, the functional $I(g)$ has the same form as before (apart from the parameter λ which is replaced by $\frac{1}{\lambda^2}$). Therefore, we obtain a general solution f of the same shape as before.

f) A variation of the original problem is obtained if the width of the top cover plate is fixed to $L > a$ before it is bended, which results in a constraint on the length of the curve f in the above figure. Which functional $I(f)$ models this new situation of maximizing the room under the cover for a fixed curve length L (the so-called 'Dido's problem')? Remember that the length of a curve f in the interval $[x_1, x_2]$ is calculated as $L = \int_{x_1}^{x_2}\sqrt{1 + (f')^2(x)}dx$.

Which necessary condition is obtained now (you do not need to calculate the total derivative)?

Verify that the solution is given by a circular arc with center (c_1, c_2):

$$(x + c_1)^2 + (f + c_2)^2 = r^2$$

Solution:

$$I(f) = -\int_{-a/2}^{a/2} f(x)dx + \lambda\int_{-a/2}^{a/2}\sqrt{1 + (f'(x))^2}dx$$

Partial derivatives of the integrand $F(x, f, f') = -f + \lambda\sqrt{1 + (f'(x))^2}$*:*

$$F_f = -1 \qquad\qquad F_{f'} = \frac{\lambda f'}{\sqrt{1 + (f')^2}}$$

Euler-Lagrange equation:

$$F_f = \frac{d}{dx}F_{f'} \qquad \Longleftrightarrow \qquad -\frac{1}{\lambda} = \frac{d}{dx}\frac{f'}{\sqrt{1 + (f')^2}}$$

Verification of the solution:

$$\begin{aligned} & f = \sqrt{r^2 - (x + c_1)^2} - c_2 \\ \Rightarrow \quad & f' = -\frac{x + c_1}{\sqrt{r^2 - (x + c_1)^2}} \end{aligned}$$

Insertion into the Euler equation:

$$\begin{aligned} -\frac{1}{\lambda} &= \frac{d}{dx}\frac{f'}{\sqrt{1 + (f')^2}} \\ &= \frac{d}{dx}\frac{-(x + c_1)}{\sqrt{r^2 - (x + c_1)^2}\sqrt{1 + \frac{(x+c_1)^2}{r^2-(x+c_1)^2}}} \\ &= \frac{d}{dx}\frac{-(x + c_1)}{\sqrt{r^2 - (x + c_1)^2 + (x + c_1)^2}} \\ &= \frac{d}{dx}\frac{-(x + c_1)}{r} \\ &= -\frac{1}{r} \end{aligned}$$

For $\lambda = r$*, the solution is verified.*