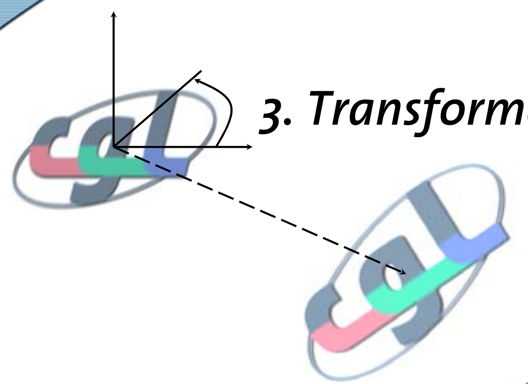


3. Transformations

Linear and Affine Mappings

- a map A: $\mathbf{x} \Rightarrow \mathbf{x'}$ is called *linear*, if

$$A(\alpha\,\mathbf{x} + \beta\,\mathbf{y}) = \alpha\,A(\mathbf{x}) + \beta\,A(\mathbf{y})$$

- a map A: $\mathbf{x} \Rightarrow \mathbf{x'}$ is called *affine*, if

$$\mathbf{x}' = A(\mathbf{x}) + \mathbf{t} = B(\mathbf{x})$$

- Linear transforms are represented by matrices, i.e. $\mathbf{Ax} = \mathbf{x'}$

2D Transforms

- Translation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} dx \\ dy \end{pmatrix} \Rightarrow \mathbf{P'} = \mathbf{P} + \mathbf{T}$$

- Scaling

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \mathbf{P'} = \mathbf{S} \times \mathbf{P}$$

2D Transforms

- Rotation of a point $\mathbf{P}$ along some angle θ

$$x' = x \cdot cos\theta - y \cdot sin\theta$$
$$y' = x \cdot sin\theta + y \cdot cos\theta$$

- Matrix form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \mathbf{P'} = \mathbf{R} \times \mathbf{P}$$

Homogenous Coordinates

- Matrix form of 3 fundamental transforms

$$P' = R \times P \qquad P' = S \times P \qquad P' = T + P$$

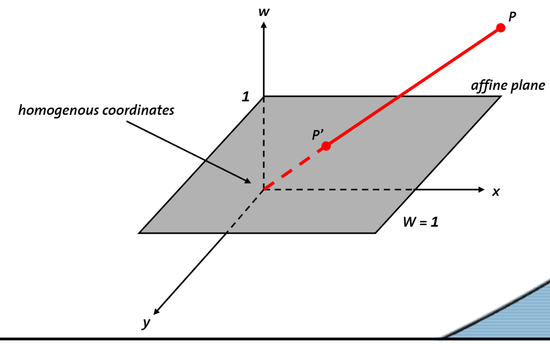
- Translation as additive component
- Take point **P** in homogenous coordinates by adding additional weight:

$$P = (x, y, W)$$

A point P in 2D has infinitely many homogenous coordinates: (2,1) can be written as (2,1,1), (4,2,2) or (-4,-2,-2)... Division by w -> projection

Homogenous Coordinates

- Point $P' = (x, y, 1)$ as line (wx, wy, w) in 3D



Translation and Scaling

- Representation by 3x3 matrix

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \times \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \Rightarrow P' = T \times P$$

- Scaling

$$S(s_x, s_y) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation and Shearing

- Rotation Matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Shears along *x*- and *y*-axis

$$SH_x = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad SH_y = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

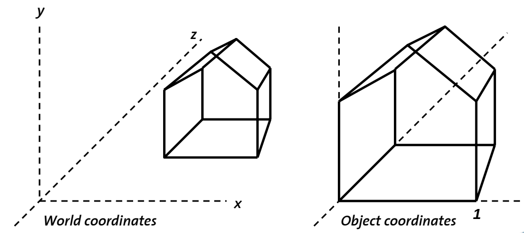
Combinations

- Stack transforms using matrix multiplication
- Example: Translation and Rotation $T \times R = \begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$
- Commutativity of Matrices ***M1***, ***M2*** given if

Matrix M1	*Matrix M2*	
Translation	*Translation*	
Rotation	*Rotation*	2D only!
Scaling	*Scaling*	
Scaling	*Rotation*	

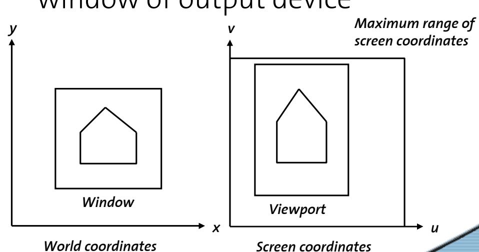
Coordinate Systems

- ***World coordinates***: reference system to describe scenery
- ***Object coordinates***: internal representation

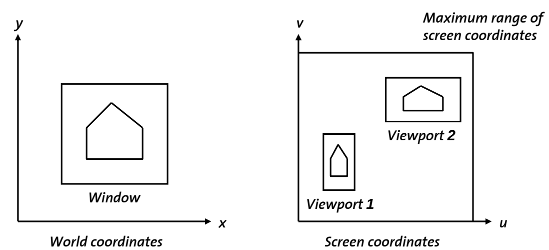


Windows and Viewports

- ***Window*** in world coordinates to represent camera
- ***Viewport*** in device coordinates to define sub-window of output device

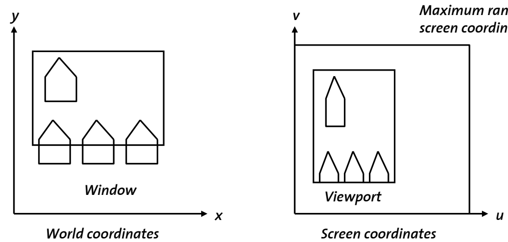


Multiple Viewports



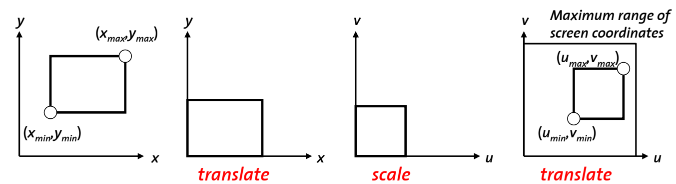
Clipping Windows

- Transform affects clipping and ***aspect ratio***



Window-Viewport Transform

- 3 step sequence of translate-scale-translate



$$M_{wv} = T(u_{min}, v_{min}) \times S\left(\frac{u_{max}-u_{min}}{x_{max}-x_{min}}, \frac{v_{max}-v_{min}}{y_{max}-y_{min}}\right) \times T(-x_{min}, -y_{min})$$

Window-Viewport Transform

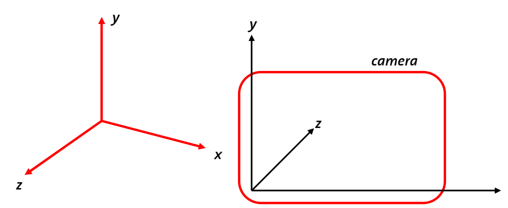
- Homogenous coordinates

$$= \begin{bmatrix} 1 & 0 & u_{min} \\ 0 & 1 & v_{min} \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \frac{u_{max}-u_{min}}{x_{max}-x_{min}} & 0 & 0 \\ 0 & \frac{v_{max}-v_{min}}{y_{max}-y_{min}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & -x_{min} \\ 0 & 1 & -y_{min} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{u_{max}-u_{min}}{x_{max}-x_{min}} & 0 & -x_{min} \cdot \frac{u_{max}-u_{min}}{x_{max}-x_{min}} + u_{min} \\ 0 & \frac{v_{max}-v_{min}}{y_{max}-y_{min}} & -y_{min} \cdot \frac{v_{max}-v_{min}}{y_{max}-y_{min}} + v_{min} \\ 0 & 0 & 1 \end{bmatrix}$$

3D Transforms

- Homogenous coordinates -> 4x4 matrices
- Project point ***p*** = (*x, y, z, w*) onto hyperplane (*x/w, y/w, z/w, 1*)
- Left-handed versus right-handed systems



Translation and Scaling

- Representation by 4x4 matrix

$$\boldsymbol{T}(t_x, t_y, t_z) = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Scaling

$$\boldsymbol{S}(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation

- Rotation matrices for x-,y-, and z-axis

$$\boldsymbol{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{R}_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about Arbitrary Axis

- Normalized axis **u**, angle θ

$$\mathbf{R}(\mathbf{u}, \theta) =$$

$$\begin{bmatrix} u_x^2 + \cos\theta(1-u_x^2) & u_x u_y(1-\cos\theta) - u_z \sin\theta & u_x u_z(1-\cos\theta) + u_y \sin\theta & 0 \\ u_x u_y(1-\cos\theta) + u_z \sin\theta & u_y^2 + \cos\theta(1-u_y^2) & u_y u_z(1-\cos\theta) + u_x \sin\theta & 0 \\ u_x u_z(1-\cos\theta) - u_y \sin\theta & u_y u_z(1-\cos\theta) + u_x \sin\theta & u_z^2 + \cos\theta(1-u_z^2) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Shear

- Shearing parallel to principal planes

$$\boldsymbol{SH}_{xy}(sh_x, sh_y) = \begin{bmatrix} 1 & 0 & sh_x & 0 \\ 0 & 1 & sh_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{SH}_{xz}(sh_x, sh_z) = \begin{bmatrix} 1 & sh_x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & sh_z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{SH}_{yz}(sh_y, sh_z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ sh_y & 1 & 0 & 0 \\ sh_z & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transform of Normal Vector

- Given a plane in implicit form

$$Ax + By + Cz + D = 0$$

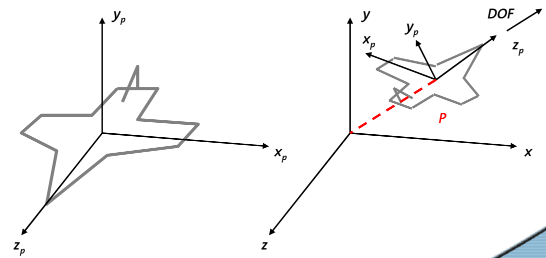
- Normal $\boldsymbol{n}$ = (A, B, C, D) and $\boldsymbol{P}$ = (x, y, z, $\boldsymbol{1}$)
- Transformed normal $\boldsymbol{n'}$ computed by

$$\boldsymbol{n}' = (\boldsymbol{M}^{-1})^T \boldsymbol{n}$$

Verification by insertion into plane equation and some algebra !

Compound 3D Transforms

- Object-to-world coordinate transform
- Example: Visual simulation



Compound 3D Transforms

1. Compute a new orthogonal basis by
   - $\mathbf{r}_1 = \boldsymbol{y} \times \boldsymbol{DOF}$
   - $\mathbf{r}_2 = \boldsymbol{DOF} \times (\boldsymbol{y} \times \boldsymbol{DOF})$ (orthogonal)
   - $\mathbf{r}_3 = \boldsymbol{DOF}$
2. Corresponding rotation matrix is

$$\mathbf{R} = \begin{bmatrix} r_{1x} & r_{2x} & r_{3x} & 0 \\ r_{1y} & r_{2y} & r_{3y} & 0 \\ r_{1z} & r_{2z} & r_{3z} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Compound 3D Transforms

2. Translate into $\boldsymbol{P}$ = (p_x, p_y, p_z, 1)

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

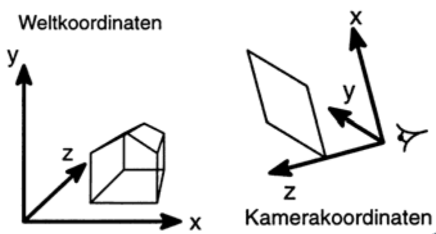
3. Compound model matrix

$$\mathbf{M} = \mathbf{TR} = \begin{bmatrix} r_{1x} & r_{2x} & r_{3x} & p_x \\ r_{1y} & r_{2y} & r_{3y} & p_y \\ r_{1z} & r_{2z} & r_{3z} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Model-View Transform

Model and View transforms are inverse to each other !

1. Left-Right transform
2. Rotate into new basis (camera)
3. Translate
4. Invert compound matrix



Modeling Transform – Forward

$$M_T \times M_R \times M_{LR} = \begin{bmatrix} 1 & 0 & 0 & P_x \\ 0 & 1 & 0 & P_y \\ 0 & 0 & 1 & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} h_{x_1} & h_{y_1} & h_{z_1} & 0 \\ h_{x_2} & h_{y_2} & h_{z_2} & 0 \\ h_{x_3} & h_{y_3} & h_{z_3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} h_{x_1} & h_{y_1} & -h_{z_1} & P_x \\ h_{x_2} & h_{y_2} & -h_{z_2} & P_y \\ h_{x_3} & h_{y_3} & -h_{z_3} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Viewing Transform – Inverse

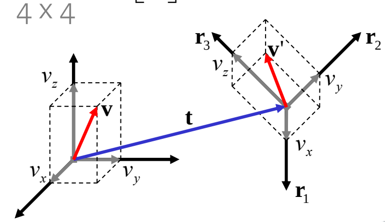
$$M_{LR}^{-1} \times M_R^{-1} \times M_T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} h_{x_1} & h_{x_2} & h_{x_3} & 0 \\ h_{y_1} & h_{y_2} & h_{y_3} & 0 \\ h_{z_1} & h_{z_2} & h_{z_3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -P_x \\ 0 & 1 & 0 & -P_y \\ 0 & 0 & 1 & -P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note the orthogonality of M_R-> transpose is inverse !

Duality of Modeling Transform and Viewing Transform (OpenGL)

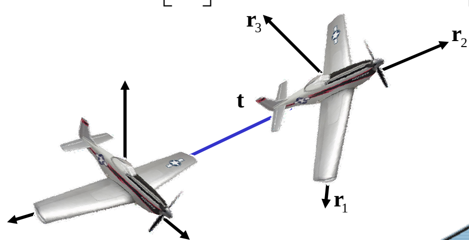
Transformation between Coordinate Systems

$$\underbrace{\begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{t} \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{4 \times 4} \cdot \begin{bmatrix} v_x \\ v_y \\ v_z \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{t} + v_x \mathbf{r}_1 + v_y \mathbf{r}_2 + v_z \mathbf{r}_3 \\ 1 \end{bmatrix}$$



Model to World

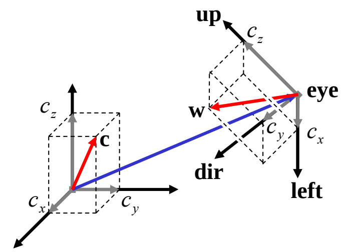
$$\begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{t} \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} m_x \\ m_y \\ m_z \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{t} + v_x\mathbf{r}_1 + v_y\mathbf{r}_2 + v_z\mathbf{r}_3 \\ 1 \end{bmatrix} = \begin{bmatrix} w_x \\ w_y \\ w_z \\ 1 \end{bmatrix}$$



World to Camera

$$\begin{bmatrix} \mathbf{dir} & \mathbf{up} & -\mathbf{left} & \mathbf{eye} \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_x \\ c_y \\ c_z \\ 1 \end{bmatrix} = \begin{bmatrix} w_x \\ w_y \\ w_z \\ 1 \end{bmatrix}$$

- Solve for **c**
- Invert Transformation



An OpenGL Example

- View Transform for entire scene

```
glMatrixMode(GL_MODELVIEW);
glLoadIdentity();
gluLookAt(0,0,2, 0,0,0, 0,1,0);  // eye, center, up
```

- Model Transform only for teapot

```
glMatrixMode(GL_MODELVIEW);
glPushMatrix();
glRotatef(30, 1.0, 1.0, 1.0);
glutSolidTeapot(0.5);
glPopMatrix();
```


Quaternions

- Elegant notion for the modeling of translation and rotation
- Compact representation
- Mathematical element with a set of operators (e.g. multiplication)
- Efficient implementation
- Widely used for animation

Definition

- A ***quaternion*** $\boldsymbol{q}$ is a "hypercomplex" number defined as: $\boldsymbol{q} = c + x\,i + y\,j + z\,k$
- *c, y, x, z* are real numbers
- *i, j, k* are imaginary
- Notation: $\boldsymbol{q} = c + \boldsymbol{u}$
- *c*: real part, $\boldsymbol{u}$: ***pure quaternion***

  $\boldsymbol{u} = x\,i + y\,j + z\,k$

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3. Transformations

Basic Properties and Operators

- Addition

  $\boldsymbol{q} + \boldsymbol{q'} = (c + c') + (x + x')i + (y + y')j + (z + z')k$
- Multiplication of complex operators

  $i^2 = j^2 = k^2 = -1$

  $i\,j = k, \quad j\,i = -k; \quad j\,k = i, \quad k\,j = -i; \quad k\,i = j, \quad i\,k = -j$
- Quaternion multiplication of $\boldsymbol{q}$ and $\boldsymbol{q'}$

  $\boldsymbol{q}\boldsymbol{q'} = (c + u)(c' + u')$

  $= (c\,c' - \boldsymbol{u} \cdot \boldsymbol{u'}) + (\boldsymbol{u} \times \boldsymbol{u'} + \langle c\,\boldsymbol{u'} \rangle + \langle c'\,\boldsymbol{u} \rangle)$

Quaternion multiplication is NOT commutative!

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3. Transformations

Basic Properties and Operators

- Inner product, scalar operator **<…>** and cross product **×** defined as:

  $\boldsymbol{u} \cdot \boldsymbol{u'} = x\,x' + y\,y' + z\,z'$

  $\langle c\,\boldsymbol{u} \rangle = c\,x\,i + c\,y\,j + c\,z\,k$

  $\boldsymbol{u} \times \boldsymbol{u'} = (y\,z' - z\,y')i + (z\,x' - x\,z')j + (x\,y' - y\,x')k$
- One-elements of addition and multiplication

  $0 = 0 + 0\,i + 0\,j + 0\,k$

  $1 = 1 + 0\,i + 0\,j + 0\,k$

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3. Transformations

Basic Properties and Operators

- Conjugate elements

  $\boldsymbol{q} = c + \boldsymbol{u} \quad ; \quad \overline{\boldsymbol{q}} = c - \boldsymbol{u}$
- Absolute

  $\boldsymbol{q}\overline{\boldsymbol{q}} = c^2 + x^2 + y^2 + z^2$

  $\boldsymbol{q}\overline{\boldsymbol{q}} = |\boldsymbol{q}|^2$
- Inverse elements of
  - addition $\quad -\boldsymbol{q} = -c - x\,i - y\,j - z\,k$
  - multiplication $\quad \boldsymbol{q}^{-1} = \frac{1}{|\boldsymbol{q}|^2}\overline{\boldsymbol{q}}$

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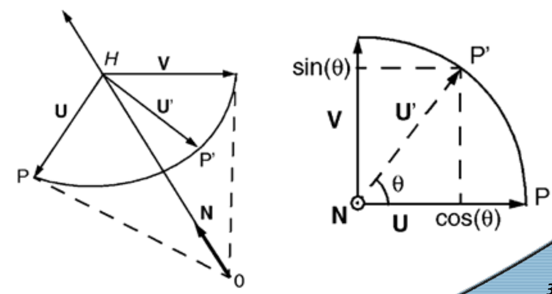
3. Transformations

Unit Quaternions

- Quaternions of length 1 are fundamental to encode transformations
- Using a unit quaternion.. $|\boldsymbol{q}|^2 = c^2 + \boldsymbol{u}\cdot\boldsymbol{u} = 1$
- .. and introducing .. $\boldsymbol{N} = [N_x, N_y, N_z]^T \quad \boldsymbol{I} = [i, j, k]^T$
- ..with.. $\boldsymbol{q} = c + \boldsymbol{u} \quad \boldsymbol{u} = s\boldsymbol{n} \quad \boldsymbol{n} = \boldsymbol{NI}$
- ..it follows that $c^2 + s^2 = 1$
- Each unit quaternion can be rewritten as
  $\boldsymbol{q} = \cos\theta + \sin\theta\,\boldsymbol{n}$
  $\Rightarrow \boldsymbol{q}\boldsymbol{q}' = \cos(\theta+\theta') + \sin(\theta+\theta')\boldsymbol{n}$

3D Rotation using Quaternions

- Rotation of a point ***P*** along an arbitrary axis ***N*** and angle θ



3D Rotation using Quaternions

- Find ***P'*** as a function of ***P*** and ***N***
  $$\boldsymbol{P}' = \cos\theta\,\boldsymbol{P} + (1-\cos\theta)\boldsymbol{N}(\boldsymbol{N}\cdot\boldsymbol{P}) + \sin\theta(\boldsymbol{N}\times\boldsymbol{P})$$
- Corresponding rotation operator ***R***$(\theta, \boldsymbol{N})$
  $$\boldsymbol{R}(\theta,\boldsymbol{N}) = \cos\theta\,\boldsymbol{I}_3 + (1-\cos\theta)\boldsymbol{N}^T\boldsymbol{N} + \sin\theta\,\boldsymbol{A}_N$$
  $$\boldsymbol{N} = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix},\quad \boldsymbol{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad \boldsymbol{A}_N = \begin{bmatrix} 0 & N_3 & -N_2 \\ -N_3 & 0 & N_1 \\ N_2 & -N_1 & 0 \end{bmatrix}$$

3D Rotation using Quaternions

- Represent point ***P*** as a pure quaternion
  $\boldsymbol{P} = (x, y, z) \qquad \boldsymbol{p} = 0 + \boldsymbol{v} = xi + yj + zk$
- Rotation using quaternion operators
  $R_q(\boldsymbol{p}) = \boldsymbol{q}\,\boldsymbol{p}\,\overline{\boldsymbol{q}} \qquad \boldsymbol{q} = c + \boldsymbol{u} = \cos\theta + \sin\theta\,\boldsymbol{n}$
- Insertion and a little algebra reveals ..
  $$R_q(\boldsymbol{p}) = \begin{bmatrix} <\cos(2\theta)\boldsymbol{v}> + \\ <(1-\cos(2\theta))(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{n} + <\sin(2\theta)(\boldsymbol{n}\times\boldsymbol{v})> \end{bmatrix}$$
- Rotation of ***P*** along axis ***N*** by angle 2θ

Points and Rotation

- Recipe: Take ***p*** and ***q*** and compute

$$p' = q p \bar{q}$$

$$q = \cos(\theta/2) + \sin(\theta/2) n$$

$$n = N_1 i + N_2 j + N_3 k$$

- Elegant implementation using operator overloading

Points and Translation

- Translation vector as a pure quaternion

$$p' = p + t$$

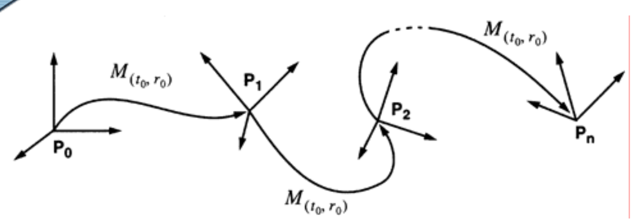
- Sequences of rotations ***r*** and translations ***t*** using a transformation operator ***M***

$$p \rightarrow p' = M_{(t,r)}(p) = r p \bar{r} + t$$

- $M_{(t,r)}$ denotes rotation-translation-operator
- $M_{(0,r)}$ describes rotation and $M_{(t,1)}$ translation

$$M_{(t,r)} = M_{(0,r)} \circ M_{(t,1)}$$

Sequencing for Animation



$$(t,r) = (t_0, r_0) \circ (t_i, r_i)$$

$$(t,r) \circ (t', r') = (t + r t' \bar{r}, r r')$$

$$(t,r) = (t_0, r_0) \circ \ldots \circ (t_i, r_i) \circ \ldots \circ (t_n, r_n)$$