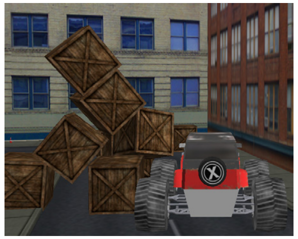
# Rigid Body Dynamics



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Seminar – Wintersemester 02/03



## Outline

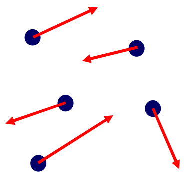






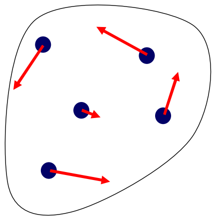
## Particle System vs. Rigid Body

**Particle System**



- 3n degrees of freedom (dof)
- interaction modeled explicitly
- system of 3n unknowns

**Rigid Body (using mesh)**

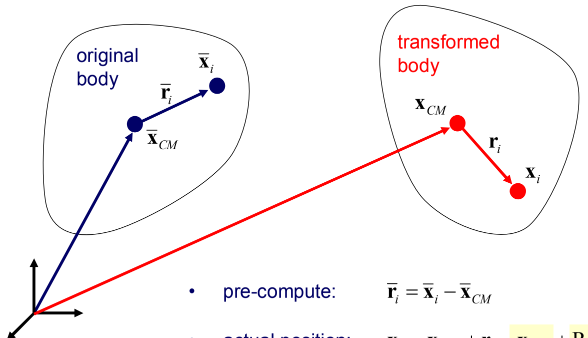


- springs with infinite stiffness
- modeled implicitly
- 6 remaining dof
- position and orientation of entire body





## Representation of a Rigid Body

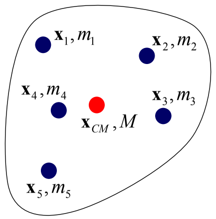



- pre-compute: $\bar{\mathbf{r}}_i = \bar{\mathbf{x}}_i - \bar{\mathbf{x}}_{CM}$
- actual position: $\mathbf{x}_i = \mathbf{x}_{CM} + \mathbf{r}_i = \mathbf{x}_{CM} + \mathrm{Rot}(\bar{\mathbf{r}}_i)$

translation ($\mathbf{x}_{CM}$), rotation ($\mathrm{Rot}(\bar{\mathbf{r}}_i)$)

## Center of Mass Definition

ETH Zurich




Definition:

$$\mathbf{x}_{\mathrm{CM}} = \frac{\sum m_i \mathbf{x}_i}{\sum m_i} = \frac{\sum m_i \mathbf{x}_i}{M}$$

$$M\mathbf{x}_{\mathrm{CM}} = \sum m_i \mathbf{x}_i$$

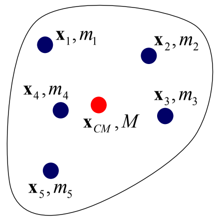
Same point on body under translation and rotation!

Continuous: $\mathbf{x}_{\mathrm{CM}} = \frac{\int \mathbf{x}\rho(x)dV}{\int \rho(x)dV}$



## Center of Mass Motivation

ETH Zurich




$$M\mathbf{x}_{\mathrm{CM}} = \sum m_i \mathbf{x}_i$$

Newton's second law:

$$\mathbf{f}_i = m_i \ddot{\mathbf{x}}_i$$

$$\mathbf{F} = \sum \mathbf{f}_i = \sum m_i \ddot{\mathbf{x}}_i = \frac{\partial^2}{\partial t^2} \sum m_i \mathbf{x}_i$$

$$= \frac{\partial^2}{\partial t^2} M\mathbf{x}_{CM} = M\ddot{\mathbf{x}}_{CM}$$

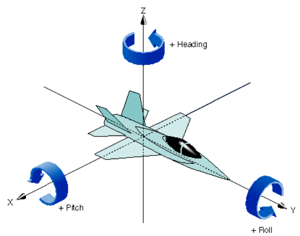
$$\mathbf{F} = M\ddot{\mathbf{x}}_{CM}$$



## Rotation in 3-d

ETH Zurich

Three Euler Angles:
- airplanes: roll, pitch, heading




- dependent on order of application
- not practical



## Rotation in 3-d

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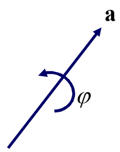
Quaternions:
- every combination of rotations can be represented by
- one rotation about one axis




$$\mathbf{q} = [s, x, y, z]$$

$$= \left[\cos(\tfrac{\varphi}{2}), \sin(\tfrac{\varphi}{2}) \cdot (a_x, a_y, a_z)\right]$$

$$\mathrm{Rot}(\mathbf{v}) = \mathbf{q} \cdot \mathbf{v} \cdot \mathbf{q}^{-1}$$

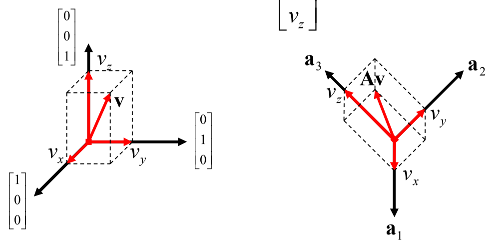
- special definition for quaternion multiplication
- one additional dof
- often used in rigid body computations

## Rotation in 3-d

ETH Zurich

Rotation matrix
- simplest way
- 6 additional dof!

$$\mathrm{Rot}(\mathbf{v}) = \mathbf{A}\mathbf{v} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] \cdot \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = v_x \mathbf{a}_1 + v_y \mathbf{a}_2 + v_z \mathbf{a}_3$$






## Rotation in 3-d

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- The columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ of $\mathbf{A}$ are the new axis!
- → A must be right handed orthonormal

$$\mathbf{A}\mathbf{A}^{-T} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] \cdot \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \mathbf{a}_3^T \end{bmatrix} = \mathbf{I}$$

$$\mathrm{Det}(\mathbf{A}) = +1$$

- actual position: $\mathbf{x}_i = \mathbf{x}_{CM} + \mathbf{r}_i = \mathbf{x}_{CM} + \mathbf{A} \cdot \bar{\mathbf{r}}_i$

translation | rotation



## Body in Motion

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Time dependent position:

$$\mathbf{x}_i(t) = \mathbf{x}_{CM}(t) + \mathbf{A}(t) \cdot \bar{\mathbf{r}}_i$$

Velocity:

$$\dot{\mathbf{x}}_i = \dot{\mathbf{x}}_{CM} + \dot{\mathbf{A}} \cdot \bar{\mathbf{r}}_i$$

linear velocity | angular velocity



## Angular Velocity in 3-d

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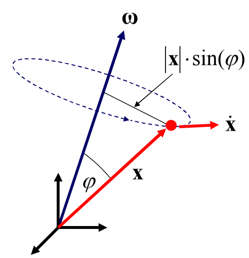
Angular velocity $\boldsymbol{\omega}$ is a vector in 3-d:
- in direction of axis of rotation
- | $\boldsymbol{\omega}$ | = angular velocity [rad/s]

$$|\dot{\mathbf{x}}| = |\boldsymbol{\omega}| \cdot r = |\boldsymbol{\omega}| \cdot |\mathbf{x}| \cdot \sin(\varphi)$$

$$\dot{\mathbf{x}} = \boldsymbol{\omega} \times \mathbf{x}$$




Define:

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \rightarrow \tilde{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

Then: $\dot{\mathbf{x}} = \tilde{\boldsymbol{\omega}} \cdot \mathbf{x}$

## Rigid Body Kinematics

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What is the relationship between $\tilde{\boldsymbol{\omega}}$ and $\dot{\mathbf{A}}$ ?

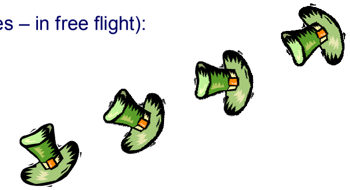
Angular velocity rotates all axis (columns of A)!

$$\dot{\mathbf{A}} = [\dot{\mathbf{a}}_1, \dot{\mathbf{a}}_2, \dot{\mathbf{a}}_2] = [\tilde{\boldsymbol{\omega}} \cdot \mathbf{a}_1, \tilde{\boldsymbol{\omega}} \cdot \mathbf{a}_2, \tilde{\boldsymbol{\omega}} \cdot \mathbf{a}_3] = \tilde{\boldsymbol{\omega}} \cdot \mathbf{A}$$

Rigid body kinematics (no forces – in free flight):

$$\dot{\mathbf{x}}_{CM} = \mathbf{v}_{\text{init}}$$
$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\text{init}}$$

$$\mathbf{x}_{CM} = \mathbf{x}_{CM} + \Delta t \cdot \dot{\mathbf{x}}_{CM}$$
$$\mathbf{A} = \mathbf{A} + \Delta t \cdot \tilde{\boldsymbol{\omega}} \cdot \mathbf{A}$$
$$\mathbf{x}_i = \mathbf{x}_{CM} + \mathbf{A} \cdot \bar{\mathbf{r}}_i$$



Physically not correct! Without forces **L** is constant not **ω**!
**ω** = **I**$^{-1}$**L** and **I** depends on **A** -> see below



## Dynamics - Let the force be with you!

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Forces change:
- Linear velocity
- Angular velocity

Linear velocity change:

$$\mathbf{F} = \sum \mathbf{f}_i = \sum m_i \ddot{\mathbf{x}}_i = \frac{\partial^2}{\partial t^2} \sum m_i \mathbf{x}_i$$
$$= \frac{\partial^2}{\partial t^2} M \mathbf{x}_{CM} = M \ddot{\mathbf{x}}_{CM}$$

$$\ddot{\mathbf{x}}_{CM} = \mathbf{F} / M = \left(\sum \mathbf{f}_i\right) / M$$



## Angular Momentum (Drehimpuls)

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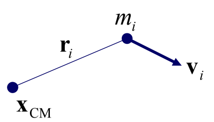


The angular momentum of particle i (w.r.t. center of mass) is:

$$\mathbf{L}_i = \mathbf{r}_i \times m_i \mathbf{v}_i = \mathbf{r}_i \times m_i \boldsymbol{\omega} \times \mathbf{r}_i$$

The total angular momentum of the body:

$$\mathbf{L} = \sum \mathbf{L}_i = \sum \mathbf{r}_i \times m_i \boldsymbol{\omega} \times \mathbf{r}_i$$
$$= \sum -m_i \tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i \boldsymbol{\omega} = \left(\sum -m_i \tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i\right) \cdot \boldsymbol{\omega}$$
$$= \mathbf{I}\boldsymbol{\omega}$$



## Inertia Tensor (Trägheitsmoment)

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We have for the total angular momentum:

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$$

where **I** is a 3x3 matrix (the inertia tensor of the body)
**I** depends on rotated configuration!

$$\mathbf{I} = \left(\sum -m_i \tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i\right)$$

Fortunately we have the relation:

$$\mathbf{I} = \mathbf{A} \cdot \bar{\mathbf{I}} \cdot \mathbf{A}^T$$

and the inertia tensor in the original body can be pre-computed:

$$\bar{\mathbf{I}} = \left(\sum -m_i \tilde{\bar{\mathbf{r}}}_i \tilde{\bar{\mathbf{r}}}_i\right)$$

## Torque (Drehmoment)

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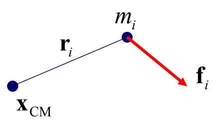


The torque of particle i (w.r.t. center of mass) is:

$$\boldsymbol{\tau}_i = \mathbf{r}_i \times \mathbf{f}_i$$

The total torque of the body:

$$\boldsymbol{\tau} = \sum \boldsymbol{\tau}_i = \sum \mathbf{r}_i \times \mathbf{f}_i$$



## Netwon's Second Law (Angular)

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Angular momentum: $\mathbf{L} = \sum \mathbf{r}_i \times m_i \mathbf{v}_i = \mathbf{I}\boldsymbol{\omega}$

Torque: $\boldsymbol{\tau} = \sum \mathbf{r}_i \times \mathbf{f}_i$

The angular version of Newton's second law reads:

$$\dot{\mathbf{L}} = \boldsymbol{\tau}$$

Tells us how the forces $\mathbf{f}_i$ change the angular velocity $\boldsymbol{\omega}$ (Euler integration):

$$\boldsymbol{\tau} = \sum \mathbf{r}_i \times \mathbf{f}_i$$

$$\mathbf{L} = \mathbf{L} + \Delta t \cdot \boldsymbol{\tau}$$

$$\boldsymbol{\omega} = \mathbf{I}^{-1}\mathbf{L}$$



## Simulation Algorithm (Euler)

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Pre-compute:

$$M \leftarrow \sum m_i$$
$$\bar{\mathbf{x}}_{CM} \leftarrow \left(\sum \bar{\mathbf{x}}_i m_i\right)/M$$
$$\bar{\mathbf{r}}_i \leftarrow \bar{\mathbf{x}}_i - \bar{\mathbf{x}}_{CM}$$
$$\bar{\mathbf{I}}^{-1} \leftarrow \left(\sum -m_i \tilde{\bar{\mathbf{r}}}_i \tilde{\bar{\mathbf{r}}}_i\right)^{-1}$$

Initialize:

$$\mathbf{x}_{CM}, \mathbf{v}_{CM}, \mathbf{A}, \mathbf{L}$$
$$\mathbf{I}^{-1} \leftarrow \mathbf{A}\bar{\mathbf{I}}^{-1}\mathbf{A}^T$$
$$\boldsymbol{\omega} \leftarrow \mathbf{I}^{-1}\mathbf{L}$$

Loop:

Sum up external forces:

$$\boldsymbol{\tau} \leftarrow \sum \mathbf{r}_i \times \mathbf{f}_i$$
$$\mathbf{F} \leftarrow \sum \mathbf{f}_i$$

Perform Euler integration step:

$$\mathbf{x}_{CM} \leftarrow \mathbf{x}_{CM} + \Delta t \cdot \mathbf{v}_{CM}$$
$$\mathbf{v}_{CM} \leftarrow \mathbf{v}_{CM} + \Delta t \cdot \mathbf{F}/M$$
$$\mathbf{A} \leftarrow \mathbf{A} + \Delta t \cdot \tilde{\boldsymbol{\omega}}\mathbf{A}$$
$$\mathbf{L} \leftarrow \mathbf{L} + \Delta t \cdot \boldsymbol{\tau}$$
$$\mathbf{I}^{-1} \leftarrow \mathbf{A}\bar{\mathbf{I}}^{-1}\mathbf{A}^T$$
$$\boldsymbol{\omega} \leftarrow \mathbf{I}^{-1}\mathbf{L}$$

Per particle quantities:

$$\mathbf{r}_i \leftarrow \mathbf{A} \cdot \bar{\mathbf{r}}_i$$
$$\mathbf{x}_i \leftarrow \mathbf{x}_{CM} + \mathbf{r}_i$$
$$\mathbf{v}_i \leftarrow \mathbf{v}_{CM} + \boldsymbol{\omega} \times \mathbf{r}_i$$



## Reorthonormalization of Rotation

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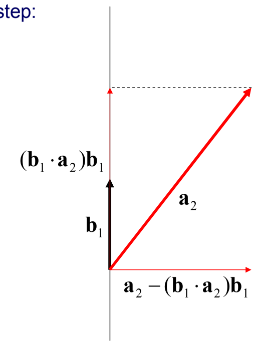
- Rotation matrix is updated at every time step:

$$\mathbf{A} \leftarrow \mathbf{A} + \Delta t \cdot \tilde{\boldsymbol{\omega}}\mathbf{A}$$

- Errors accumulate
- A is not orthonormal anymore
- Use Gram-Schmidt Orthogonalization

$$\mathbf{b}_1 = \mathbf{a}_1 / |\mathbf{a}_1|$$
$$\mathbf{b}_2 = \mathbf{a}_2 - (\mathbf{b}_1 \cdot \mathbf{a}_2)\mathbf{b}_1$$
$$\mathbf{b}_2 = \mathbf{b}_2 / |\mathbf{b}_2|$$
$$\mathbf{b}_3 = \mathbf{a}_3 - (\mathbf{b}_1 \cdot \mathbf{a}_3)\mathbf{b}_1 - (\mathbf{b}_2 \cdot \mathbf{a}_3)\mathbf{b}_2$$
$$\mathbf{b}_3 = \mathbf{b}_3 / |\mathbf{b}_3|$$




- better: $\mathbf{b}_3 = \mathbf{b}_1 \times \mathbf{b}_2$

## Force vs. Torque Puzzle

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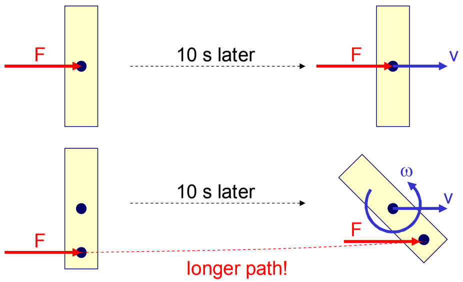
Is force being considered twice?
- To accelerate center of mass
- To cause the body to spin

$$\boldsymbol{\tau} \leftarrow \sum \mathbf{r}_i \times \mathbf{f}_i$$

$$\mathbf{F} \leftarrow \sum \mathbf{f}_i$$




$$E = \frac{1}{2} M \mathbf{v}^T \mathbf{v}$$

$$E = \frac{1}{2} M \mathbf{v}^T \mathbf{v} + \frac{1}{2} \boldsymbol{\omega}^T \mathbf{I} \boldsymbol{\omega}$$



## Non-Penetration

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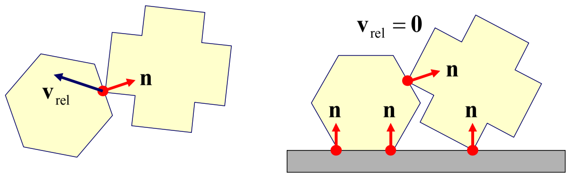
- Detect collisions (see Matthias Teschner's slides)
- Avoid penetration
  - change time step or
  - push body back
- Compute collision response
  - Colliding contacts ("easy")
  - Resting contacts (very hard!)






## Colliding Contacts

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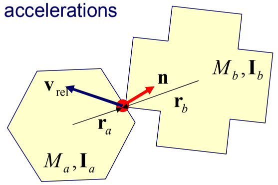
- Force driven
  - Penetration cause forces
  - Late, slow, easy to compute
- Impulse driven
  - Manipulation of velocities instead of accelerations
  - Fast, more difficult to compute
  - Impulse J changes body state:

$$\Delta \mathbf{v}_{CM} = \mathbf{J} / M$$
$$\Delta \mathbf{L} = (\mathbf{x}_{\text{impact}} - \mathbf{x}_{CM}) \times \mathbf{J}$$
$$\mathbf{J} = j\mathbf{n}$$




$$j = \frac{-(1+e)v_{\text{rel}}}{\frac{1}{M_a} + \frac{1}{M_b} + \left[ (\mathbf{I}_a^{-1}(\mathbf{r}_a \times \mathbf{n})) \times \mathbf{r}_a + (\mathbf{I}_b^{-1}(\mathbf{r}_b \times \mathbf{n})) \times \mathbf{r}_b \right] \cdot \mathbf{n}}$$



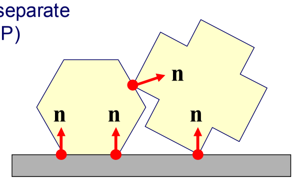
## Resting Contacts

ETH Zurich

- Find all collisions with

  $|\mathbf{v}_{rel}| < \varepsilon$

- Solve for all contact forces simultaneously such that for each contact force $\mathbf{f}_i$
  - $\mathbf{f}_i$ is strong enough that bodies are not pushed towards one another
  - $\mathbf{f}_i$ must be repulsive only (not glue like)
  - $\mathbf{f}_i$ is zero if the bodies begin to separate
- Linear complementarity problem (LCP)
- Special case of a (QPP)
  Quadratic Programming Problem!



# Web Sites

ETH Zurich

- **Andrew Witkin, David Baraff: *Physically Based Modeling: Principles and Practice* (Online Siggraph '97 Course notes)**

  www-2.cs.cmu.edu/~baraff/sigcourse/

- **Chris Hecker: *Rigid Body Dynamics***

  www.d6.com/users/checker/dynamics.htm

- **NovodeX Rigid Body SDK & Demos**

  www.novodex.com