Interactive Simulation of Elasto-Plastic Materials using the Finite Element Method

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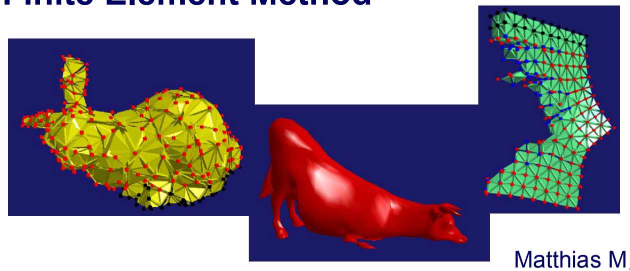

Matthias M¸ller
Seminar ñ Wintersemester 02/03

Outline

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FEM vs. Mass-Spring
Stiffness
- The Stiffness Matrix
- Static/Dynamic Deformation

Continuum Mechanics and FEM
- Strain and Stress Tensors
- Continuous PDEís
- FEM Discretization

Plasticity
- Plastic Strain
- Update Rules

Fracture
- Principal Stresses
- Crack Computation

Mass-Spring vs. FEM

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1. Discretization of an object into **mass points**
2. Representation of forces between mass points with **springs**
3. Computation of the dynamics

➡ deformable mass-spring system

1. Discretization of an object into elements (tetrahedra)
2. Discretization of continuous energy equations into algebraic equations for forces acting at vertices
3. Computation of the dynamics

➡ deformable FEM system

Pros and Cons of FEM

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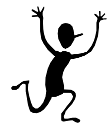
Pros:
- ï No individual spring constants needed (only 2 known material parameters E,ν)
- ï No inversion problems (inverted tetrahedra produce forces)
- ï Stress and strain tensors allow
  - ï fracture and
  - ï plasticity simulations



Cons:
- ï (Pre-)compute stiffness matrix
- ï Store stiffness matrix (3x3) per edge
- ï Store original **and** actual positions of vertices

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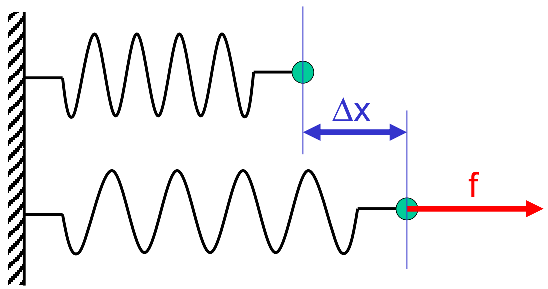
One-dimensional Spring

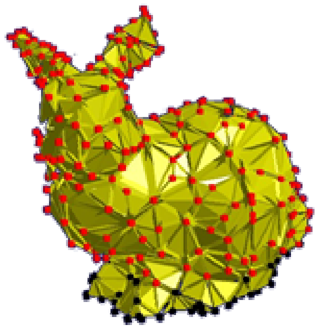


$$f = k \cdot \Delta x$$

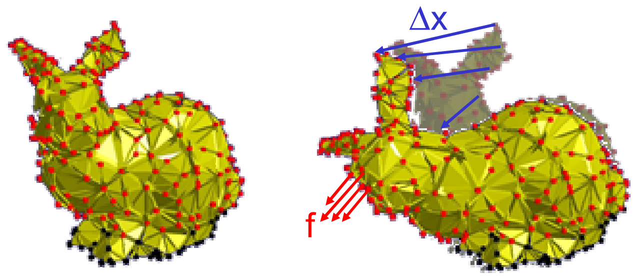
Three-dimensional Object

Finite Element Mesh
ï 903 tetrahedra
ï 393 vertices
ï 3 x 393 = 1179 dof.

$$\Delta \mathbf{x} = \begin{bmatrix} \Delta x_{1x} \\ \Delta x_{1y} \\ \Delta x_{1z} \\ \ldots \\ \Delta x_{nx} \\ \Delta x_{ny} \\ \Delta x_{nz} \end{bmatrix} \quad \mathbf{f} = \begin{bmatrix} f_{1x} \\ f_{1y} \\ f_{1z} \\ \ldots \\ f_{nx} \\ f_{ny} \\ f_{nz} \end{bmatrix}$$



Three-dimensional Object



$\mathbf{f}_{el} = K \cdot \Delta\mathbf{x}$ (Stiffness Matix $K \in \mathbf{R}^{3n\times 3n}$)

$\mathbf{f}_{el} = F(\Delta\mathbf{x})$ (Function $F : \mathbf{R}^{3n} \rightarrow \mathbf{R}^{3n}$)

Static Deformation

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$f_{ext} = f_{el} = K \cdot \Delta x$

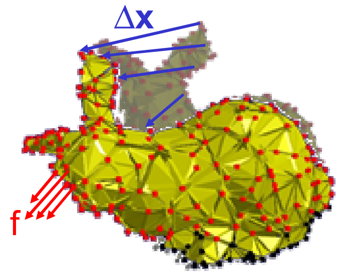
$\Delta x = K^{-1} \cdot f_{ext}$

Solve linear system
(Conjugate Gradients)

$f_{ext} = f_{el} = F(\Delta x)$

$\Delta x = F^{-1}(f_{ext})$

Solve non-linear system
(Newton-Raphson - generalized Newton-Method)



Dynamic Deformation

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$M\ddot{x} + C\dot{x} + K\Delta x = f_{ext}$

- Coupled system of 3n linear ODEs
- Explicit integration: No solver needed
- Implicit integration: Linear solver per time step

$M\ddot{x} + C\dot{x} + F(\Delta x) = f_{ext}$

- Coupled system of 3n non-linear ODEs
- Explicit integration: No solver needed
- Implicit integration: Linearize at every time step: K = dF/dx

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Continuous Elasticity 1-d

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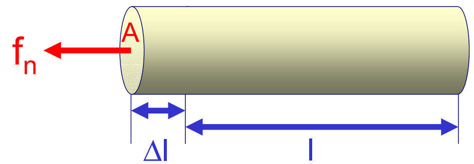

$f_n/A = E\ \Delta l/l$

stress σ [N/m²] (Normal-Spannung)

strain ε [1] (Dehnung)

Elasticity (Young's) Modulus [N/m²]
Metal: ~10^{11} N/m²
Soft material: ~10^{6} N/m²

Continuous Elasticity 3-d

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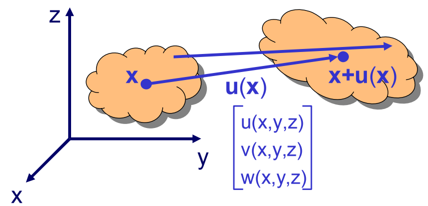
Deformation:



- Continuous 3-d vector field u: $\mathbf{R}^3 \rightarrow \mathbf{R}^3$
- Defined within undeformed object

Linear 3-d Strain

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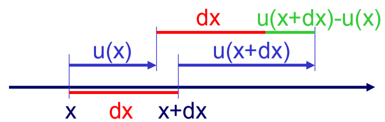
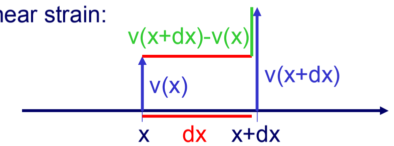
normal strain in x - direction:



$\varepsilon_x = \frac{u(x+dx)-u(x)}{dx}$

$= \partial/_{\partial x} u = u_x$

shear strain:



$\gamma_{xy} = \frac{v(x+dx)-v(x)}{dx}$

$= \partial/_{\partial x} v = v_x$

Linear 3-d Strain

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Linear strain tensor:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(x,y,z) = \begin{bmatrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_y & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_z \end{bmatrix} = \begin{bmatrix} u_x & u_y + v_x & u_z + w_x \\ v_x + u_y & v_y & v_z + w_y \\ w_x + u_z & w_y + v_z & w_z \end{bmatrix}$$

Symmetric, 3x3 matrix → 6 vector:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ \partial/\partial y & \partial/\partial x & 0 \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Non-Linear 3-d Strain

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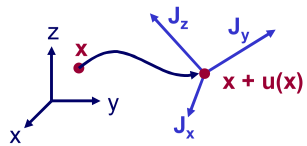
Green-Saint-Venant strain tensor:

- Transformation we use: $\mathbf{x} \rightarrow \mathbf{x} + \mathbf{u}(\mathbf{x})$:
- Use Jacobian of transformation:

$$\mathbf{J} = (\mathbf{J}_x, \mathbf{J}_y, \mathbf{J}_z) = \left(\begin{bmatrix} \partial/\partial x(x+u) \\ \partial/\partial x(y+v) \\ \partial/\partial x(z+w) \end{bmatrix}, \begin{bmatrix} \partial/\partial y(x+u) \\ \partial/\partial y(y+v) \\ \partial/\partial y(z+w) \end{bmatrix}, \begin{bmatrix} \partial/\partial z(x+u) \\ \partial/\partial z(y+v) \\ \partial/\partial z(z+w) \end{bmatrix} \right)$$

$$\boldsymbol{\varepsilon}_{\text{Green}} = \mathbf{J}^{\mathrm{T}}\mathbf{J} - \mathbf{I}$$

- Interpretation:

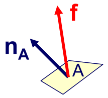


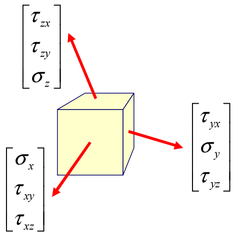
3-d Stress

Stress is force per (oriented) area:

$$\boldsymbol{\sigma} = \frac{d\mathbf{f}}{d\mathbf{A}} = \frac{d\mathbf{f}}{dA \cdot \mathbf{n}_A}$$

$$\frac{d\mathbf{f}}{dA} = \boldsymbol{\sigma} \cdot \mathbf{n}_A = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \cdot \mathbf{n}_A$$





Constitutive Relation (isotropic)

The stress tensor is symmetric, 3x3 matrix → 6 vector:

Hookeës law: $\boldsymbol{\sigma} = \mathbf{E}\,\boldsymbol{\varepsilon}$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} (1-\nu)c & \nu c & \nu c & & & \\ \nu c & (1-\nu)c & \nu c & & 0 & \\ \nu c & \nu c & (1-\nu)c & & & \\ & & & G & & \\ & 0 & & & G & \\ & & & & & G \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

$$c = \frac{E}{(1+\nu)(1-2\nu)}, G = \frac{E}{2(1+\nu)}$$

Only two scalar parameters:

E: Youngís modulus, ν: Poisson ratio

Putting it all together

Given **u**(**x**) we can compute

- strain ε(**x**) and
- stress **σ**(**x**) = E ε(**x**)

at every point **x** within the object.

→ Find **u**(**x**) such that corresponding stresses **σ**(**x**) are in balance with external forces **f**(**x**) everywhere within object:

$$\begin{aligned} \sigma_{x,x} + \tau_{xy,y} + \tau_{xz,z} + f_x &= 0 \\ \tau_{yx,x} + \sigma_{y,y} + \tau_{yz,z} + f_y &= 0 \\ \tau_{zx,x} + \tau_{zy,y} + \sigma_{z,z} + f_z &= 0 \end{aligned}$$

- Strong formulation
- Coupled system of partial differential equations!

Energy Formulation

- Energy U is a scalar
- U at point **x** is given by Ñdisplacement × forceì:

$$U_{\text{elastic}} = \frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\sigma} = \frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{E}\,\boldsymbol{\varepsilon}$$

- The total Energy of the deformed body:

$$U_{\text{body}} = \int_{\text{body}} \left(\frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{E}\,\boldsymbol{\varepsilon} - \mathrm{u} \cdot \mathrm{f} \right) dV$$

- Given **f**, **E** we can compute U_{body} for any **u**(**x**)
- Find **u**(**x**) such that U_{body} is a minimum (δU = 0)

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Finite Element Formulation

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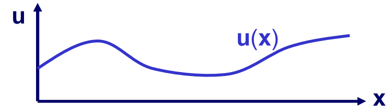
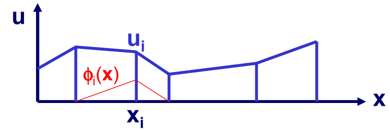
- So far we looked for a continuous field **u**(**x**)



- Now we look for $\mathbf{u}_1, \mathbf{u}_2, .. \mathbf{u}_n$ at fixed locations: $\mathbf{x}_1, \mathbf{x}_2, .. \mathbf{x}_n$
- and interpolate **u**(**x**) with fixed basis functions: $\mathbf{u}(\mathbf{x}) \approx \sum_i \mathbf{u}_i\, \phi_i(\mathbf{x})$



Linear Displacement Tetrahedron

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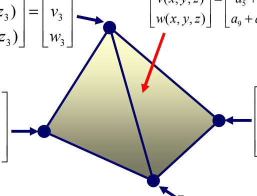
- 12 unknowns (a_1,..a_{12}), 12 equations
- u_i, v_i, w_i are variables, x_i, y_i, z_i are given numbers

$$\begin{bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{bmatrix} = \begin{bmatrix} a_1 + a_2 x + a_3 y + a_4 z \\ a_5 + a_6 x + a_7 y + a_8 z \\ a_9 + a_{10} x + a_{11} y + a_{12} z \end{bmatrix}$$

$$\begin{bmatrix} u(x_3,y_3,z_3) \\ v(x_3,y_3,z_3) \\ w(x_3,y_3,z_3) \end{bmatrix} = \begin{bmatrix} u_3 \\ v_3 \\ w_3 \end{bmatrix}$$



$$\begin{bmatrix} u(x_1,y_1,z_1) \\ v(x_1,y_1,z_1) \\ w(x_1,y_1,z_1) \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} \qquad \begin{bmatrix} u(x_4,y_4,z_4) \\ v(x_4,y_4,z_4) \\ w(x_4,y_4,z_4) \end{bmatrix} = \begin{bmatrix} u_4 \\ v_4 \\ w_4 \end{bmatrix}$$

$$\begin{bmatrix} u(x_2,y_2,z_2) \\ v(x_2,y_2,z_2) \\ w(x_2,y_2,z_2) \end{bmatrix} = \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix}$$

Displacements

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The displacement function **u**(**x**) can be expressed as
- a matrix of basis functions **H**(**x**) times
- a vector of displacements:

$$\mathbf{u}(\mathbf{x}) = \mathbf{H}(\mathbf{x}) \cdot \mathbf{\hat{u}}$$

$$\begin{bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{bmatrix}$$

Strain

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Linear displacements yield constant strain:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ \partial/\partial y & \partial/\partial x & 0 \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \end{bmatrix} \mathbf{H}(\mathbf{x}) \cdot \grave{\mathbf{u}} = \mathbf{B} \cdot \grave{\mathbf{u}}$$

ï Matrix B ∈ $\mathbf{R}^{6x12}$ is constant (independent of x,y,z)

ï B depends on the original geometry of the tetrahedron only

Stress and Energy

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Stress as a function of the displacements:

$$\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon} = \mathbf{E}\mathbf{B} \cdot \grave{\mathbf{u}}$$

Energy as a function of the displacements:

$$U_{\text{element}} = \int_{\text{element}} \left(\frac{1}{2} \boldsymbol{\varepsilon}^{\mathrm{T}} \mathbf{E}\, \boldsymbol{\varepsilon} \right) dV$$

$$= \int_{\text{element}} \frac{1}{2} \grave{\mathbf{u}}^{\mathrm{T}} \mathbf{B}^{\mathrm{T}} \mathbf{E}\mathbf{B}\, \grave{\mathbf{u}}\, dV$$

$$= \frac{1}{2} \grave{\mathbf{u}}^{\mathrm{T}} \left[\int_{\text{element}} \mathbf{B}^{\mathrm{T}} \mathbf{E}\mathbf{B}\, dV \right] \grave{\mathbf{u}} = \frac{1}{2} \grave{\mathbf{u}}^{\mathrm{T}} \left[V_{\text{element}}\, \mathbf{B}^{\mathrm{T}} \mathbf{E}\mathbf{B} \right] \grave{\mathbf{u}}$$

$$= \frac{1}{2} \grave{\mathbf{u}}^{\mathrm{T}} \mathbf{K}\, \grave{\mathbf{u}}$$

Stiffness Matrix

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$$U_{\text{body}} = \frac{1}{2} \grave{\mathbf{u}}^{\mathrm{T}} \mathbf{K}\, \grave{\mathbf{u}}$$

Forces are the derivatives of the energy with respect to the degrees of freedom:

$$\frac{\partial U_{\text{body}}}{\partial \grave{\mathbf{u}}} = \mathbf{K}\, \grave{\mathbf{u}} = \mathbf{f}$$

The matrix K ∈ $\mathbf{R}^{12x12}$ is the stiffness matrix of the element!

$$\mathbf{K} = \int_{\text{element}} \mathbf{B}^{\mathrm{T}} \mathbf{E}\mathbf{B}\, dV = V_{\text{element}}\, \mathbf{B}^{\mathrm{T}} \mathbf{E}\mathbf{B}$$

Assembly of elements

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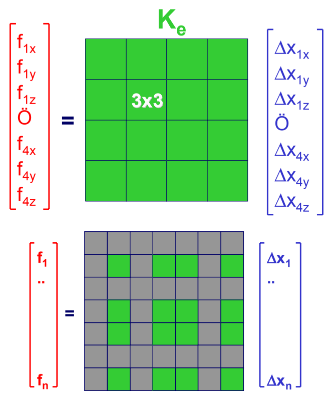
Single element:

Entire body:



Implementation

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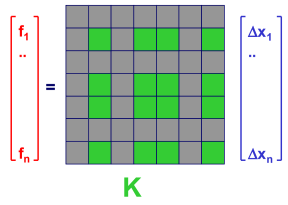

- ï K is sparse
- ï 3x3 block at (3i,3j) describes how $\Delta \mathbf{x}_j$ influences $\mathbf{f}_i$
- ï every vertex stores adjacency list of (3x3-matrix, vertex-reference) pairs

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Plastic Strain

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An element is under strain ε· due to displacements ° :

$\varepsilon\cdot = [\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}]^T = B\cdot°$

A plastic element Ñstoresì strain in a state variable:

$\varepsilon_{plastic}$

The elastic strain (that causes internal forces) is now:

$\varepsilon_{elastic} = \varepsilon\cdot - \varepsilon_{plastic}$

→ No internal forces are present when $\varepsilon\cdot = \varepsilon_{plastic}$
Might be for ° ≠ **0**!

Plastic Update Rules

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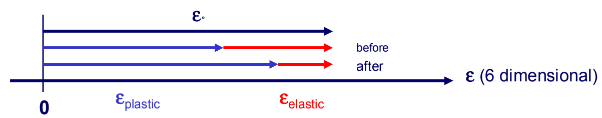
Initialization: $\varepsilon_{plastic} = \mathbf{0}$

Update rule (every time step):

- ï Compute $\varepsilon\cdot = B\cdot°$ from actual displacements
- ï Compute $\varepsilon_{elastic} = \varepsilon\cdot - \varepsilon_{plastic}$
- ï if $||\varepsilon_{elastic}||$ > yield then $\varepsilon_{plastic} = \varepsilon_{plastic}$ + creep · $\varepsilon_{elastic}$
- ï if $||\varepsilon_{plastic}||$ > max then $\varepsilon_{plastic} = \varepsilon_{plastic}$ · max / $||\varepsilon_{plastic}||$



Implementation

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Since

$\varepsilon = B \cdot °$

the displacements that correpsond to the plastic strain are:

$°_{plastic} = B^{-1} \cdot \varepsilon_{plastic}$

and the correpsonding forces are:

$\mathbf{f}_{plastic} = KB^{-1} \cdot \varepsilon_{plastic} = [vB^TEB] B^{-1} \cdot \varepsilon_{plastic} = vB^TE \cdot \varepsilon_{plastic}$

→ add plastic forces to external forces

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Fracture criterion

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- Break if internal elastic force exeeds threshold
- stress is a tensor
- the force w.r.t. normal $\mathbf{n}_A$ is:

$$\frac{d\mathbf{f}}{dA} = \boldsymbol{\sigma} \cdot \mathbf{n}_A = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \cdot \mathbf{n}_A$$

- Find $\mathbf{n}_{max}$ such that d**f**/dA is maximal!
- $\mathbf{n}_{max}$ is direction of maximal tensile stress

Principal Stresses

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- The stress tensor σ is symmetric
- → there is a rotation matrix R such that

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} = \mathbf{R}^T \begin{bmatrix} \sigma_x^p & 0 & 0 \\ 0 & \sigma_y^p & 0 \\ 0 & 0 & \sigma_z^p \end{bmatrix} \mathbf{R}$$

- the diagonal entries are the eigenvalues of σ
- the columns of R are the corresponing eigenvectors
- there is always a rotated coordinate system where σ is diagonal!

Principal Stresses

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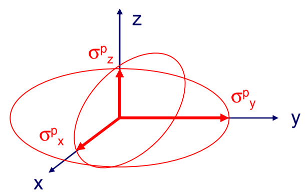

- the σ^p are the principal (extremal) stresses
- → find maximal eigenvalue of $\boldsymbol{\sigma}$
- corresponding eigenvector is the direction of maximal stress

Crack computation

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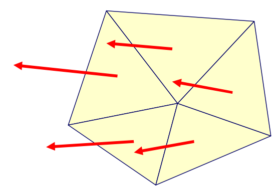
- for all elements: compute maximal tensile stress σ^p_{max}



Crack computation

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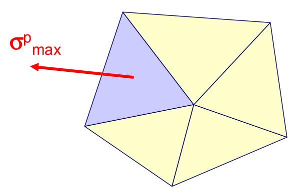
- for all elements: compute maximal tensile stress σ^p_{max}
- if σ^p_{max} exceeds yield stress



Crack computation

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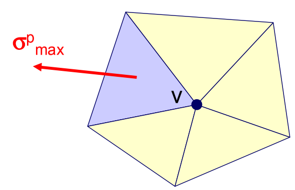
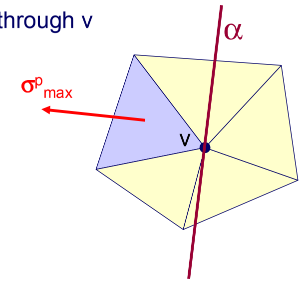
- for all elements: compute maximal tensile stress σ^p_{max}
- if σ^p_{max} exceeds yield stress
- select a vertex v (crack tip / random)



Crack computation

ETH Zurich

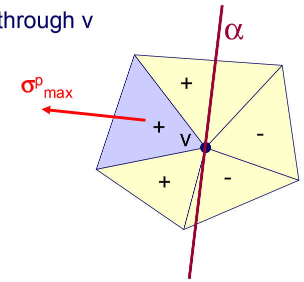
- for all elements: compute maximal tensile stress σ^p_{max}
- if σ^p_{max} exceeds yield stress
- select a vertex v (crack tip / random)
- set plane α normal σ^p_{max} to through v



Crack computation

ETH Zurich

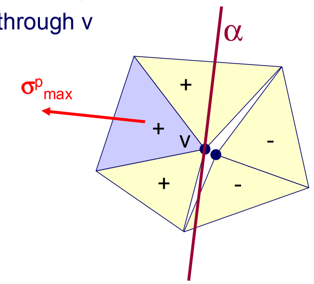
- for all elements: compute maximal tensile stress σ^p_{max}
- if σ^p_{max} exceeds yield stress
- select a vertex v (crack tip / random)
- set plane α normal σ^p_{max} to through v
- mark tetras w.r.t. α



Crack computation

ETH Zurich

- for all elements: compute maximal tensile stress σ^p_{max}
- if σ^p_{max} exceeds yield stress
- select a vertex v (crack tip / random)
- set plane α normal σ^p_{max} to through v
- mark tetras w.r.t. α
- split vertex



The End

ETH Zurich

Thank you for your attention!